Elliptical Stable Distributions

Vadym Omelchenko[1]

Abstract. The elliptical stable distributions represent a symmetric subfamily of the stable distributions. Their advantage contrary to the general stable distributions consists in their easy-to-use property and the highest resemblance to the normal distribution. They enable an easy representation of the dependence structure of the margins by means of a matrix Q the same as in case of the normal distribution. In general, the dependence structure between margins is given in form of a spectral measure which can be even continuous. The computations and approximations require so much time that it justifies the fact that many practitioners avoid using general stable distributions. The general stable distributions possess so many additional properties that they barely take after the multivariate normal distribution. But the multi-variate elliptical stable distributions can be easily simulated and the estimation of their parameters can be obtained by methods whose preciseness is almost the same as the one of the maximum likelihood methodology.

Keywords: Stable Distribution, Elliptical stable distributions, Maximum Likelihood Projections Estimators.

JEL classification: C44
AMS classification: 90C15

Introduction

The stable distributions represent a flexible parametric family of distributions capable of generalizing the normal distribution and fitting larger amounts of data preserving convolution and limit properties of the normal distribution. Under normality assumption, many real phenomena that we can often observe in practice are almost impossible, in other words their probability is so low that from practical point of view their frequency of appearance should be once in millions or even more years. That is why the practitioners and theorists try to replace the normal distribution by another distributions with heavy tails to make the model closer to reality. The stable distributions represent an ideal candidate because under the stability assumption, it is not necessary to completely rule out normality. Sometimes it happens that an unexpected jump or drop occurs not being accompanied with any important economic news nor any important technical signal, e.g. the October crush. Such situations can be explained only by appearance of an outlier in the model ruling the prices and it is a next argument favoring using the heavy tailed distributions. The fact that most of time the prices spend in flat justifies using the symmetric stable distributions which represent a sub-family of the elliptical stable distributions.
Definition. The random variable X has a univariate elliptical stable distribution if its characteristic function is of the form:

$$\psi(t) = \exp\left(i\mu t\right)\exp\left(-\sigma^{\alpha}|t|^{\alpha}\right)$$

where $\alpha \in (0,2]$, $\mu \in (-\infty,\infty)$ and $\sigma > 0$. Note that if $\alpha = 2$ we have a characteristic function of the normal distribution. If $\alpha < 2$ then any moment EX^a with $a \geq \alpha$ is infinite. If $a < \alpha$ then EX^a is finite. Hence follows that if $\alpha < 2$ then the variance var$X$ is infinite.
Definition. The multivariate elliptical stable distribution is the distribution whose characteristic function is of the form:

$$\psi(\mathbf{t}) = \exp(i \cdot \mathbf{t}^T \mu)\exp\left(-\left|\mathbf{t}^T\mathbf{Q}\mathbf{t}\right|^{\alpha/2}\right)$$

where μ is the mean vector provided that $\alpha > 1$ and $\mathbf{Q}$ is the matrix determining the dependence structure between the margins. $\mathbf{Q}$ is a positively definite matrix and in case of α equal to 2 we have the

[1]Charles University in Prage/Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, e-mail: vadim224@yahoo.com

multivariate normal distribution whose covariance matrix is $\mathbf{Q}$. In financial practice α is always larger than 1 therefore, μ is a mean vector. A very important property of the stable distribution is the fact the linear combination of the random variables with the same α has also a stable distribution with the same α parameter. In other words, if X_1, X_2,...,X_n, X are i.i.d. elliptical stable distributions with zero mean, i.e. $X_i \sim \psi(\mathbf{t}) = \exp\left(-\left|\mathbf{t}^T\mathbf{Q}\mathbf{t}\right|\right)$, $i = 1, 2, ...$ then

$$X_1 + X_2 + ... + X_n =_d \frac{1}{n^{1/\alpha}} X$$

This property can be used as a statistical test of stability by exploring if this convolutional property holds. A general univariate stable distribution is denoted as $S_\alpha(\sigma, \beta, \mu)$, where μ and β are location and skewness parameters respectively. The elliptical stable distributions are a sub-family of the general stable distributions of the form $S_\alpha(\sigma, 0, 0)$.

1 Simulation of the stable elliptical distributions

If $Z \sim$ r.v. with ch.f. $\exp(-\{t^T\mathbf{Q}t\}^{\alpha/2})$ then

$$\mathbf{Z} = \sqrt{s}\mathbf{G}$$

where $s \sim S_{\alpha/2}\left(\left(\cos\left(\frac{\pi\alpha}{4}\right)\right)^{2/\alpha}, 1, 0\right)$ and $\mathbf{G} \sim N(\mathbf{0}, \mathbf{Q})$. $\mathbf{G} = \mathbf{C}^T\mathbf{Y}$, $\mathbf{C} \cdot \mathbf{C}^T = \mathbf{Q}$, $\mathbf{Y} = (Y_1, Y_2, ..., Y_n)^T$, $Y_i \sim N(0, 1)$, $i = 1, 2, .., n$. [5] If we want to simulate a sample from $S_\alpha(1, \beta, 0)$ we can do it as follows:

$$X = S_{\alpha,\beta} \cdot \frac{\sin\{\alpha(V + B_{\alpha,\beta})\}}{\{\cos(V)\}^\alpha} \left[\frac{\cos\{V - \alpha(V + B_{\alpha,\beta})\}}{W}\right]^{\frac{1-\alpha}{\alpha}},$$

where

$$B_{\alpha,\beta} = \frac{\arctan\left(\beta \tan\frac{\pi\alpha}{2}\right)}{\alpha}, \quad S_{\alpha,\beta} = \left\{1 + \beta \tan^2\left(\frac{\pi\alpha}{2}\right)\right\}^{1/2\alpha};$$

$W \sim \exp(1)$, $V \sim U(-\pi/2, \pi/2)$. [2] If $X \sim S_\alpha(1, \beta, 0)$ then for all $\sigma > 0$ and $\nu \in R$ we have $Y = \sigma X + \mu \sim S_\alpha(\sigma, \beta, \mu)$. Thus, we can simulate any elliptical stable distribution. The simulating of a general multivariate stable distribution is a more complicated task and requires numerical techniques to conduct it. That is another reason of preference of this sub-family.

2 Parameter estimation of the multivariate elliptical stable distributions

If we want to estimate parameters of the multivariate elliptical stable distribution with parameters α and $\mathbf{Q}$ we can do it in two phases:

- Estimation of α parameter
- Estimation of the matrix $\mathbf{Q}$

The former task can be conducted from the observations of the marginal distributions because all the margins have the same α parameter. Having estimated α, we can put its estimate $\hat{\alpha}$ into the formula of the characteristic function to estimate the matrix $\mathbf{Q}$. Both estimation procedures will be conducted based on the methodology of projections.
Remark. Due to the fact that the estimation of the tail index α is the most important we will conduct the estimation of α in univariate case under assumption that $\sigma = 1$. But it is easy to transform this method to the general case and this point of issue is considered in the part about estimation of the $\mathbf{Q}$ parameter.

2.1 Estimation of the α parameter.

If X has a d-variate elliptical stable distribution with the parameters α and $\mathbf{Q}$ then any of its margins X_i, $i = 1, 2, .., d$ has a stable distribution $S_\alpha(\sigma_i, 0, 0)$ with the characteristic function of the form:

$$\psi(t) = \exp(-\sigma_i^\alpha |t|^\alpha)$$

The idea of this method is based on the properties of the maximum likelihood estimator:
If $p(x, \alpha)$ is a density function of the stable distribution then:

$$I(\alpha) = \int_{-\infty}^{\infty} J^2(x, \alpha) p(x, \alpha) dx, \quad J(x, \alpha) = \frac{\left(\frac{\partial p(x,\alpha)}{\partial \alpha}\right)}{p(x, \alpha)}, \quad \hat{\alpha}_{ML} = \left\{ \alpha : \sum_{j=1}^{n} J(X_j, \alpha) = 0 \right\}$$

where $I(\alpha)$ is the Fisher information and $X_1, X_2, ..., X_n$ is the vector of observations. The methodology of projections enables to approximate the function $J(x, \alpha)$ which will enable us to obtain the estimates of almost the same preciseness as the ML-estimators and to calculate the Fisher information. We want to project the function $J(x, \alpha)$ to the space $\{1, \exp(it_1 x), \exp(it_2 x), ..., \exp(it_k x)\}$, (The first to use such methodology was Kagan [4], but he used projections with powers. But the powers cannot be used for the stable distributions due to their infinity of the variance) i.e. to represent its approximation in form $J_k(x, \alpha)$ where

$$J_k(x, \alpha) = \sum_{j=0}^{k} \exp(it_j x) = \sum_{j=0}^{k} \cos(it_j x) + i \sum_{j=0}^{k} \sin(it_j x)$$

where a_j $j = 0, 2..., k$ are unknown parameters and $t_0, t_1, ..., t_k$ are known points in the vicinity of 0. One of possible choices of t_j, $j = 0, 1, ..., k$ is $t_j = j/k$. We project to the space with scalar product given as follows:

$$X \sim S_\alpha(1, 0, 0), \quad \langle \exp(it_m X), \exp(it_n X) \rangle = E \exp(it_m X) \exp(it_n X) =$$
$$= E \exp(iX(t_m + t_n)) = \int_{-\infty}^{\infty} p(x, \alpha) \exp(ix(t_m + t_n)) dx = \exp\left(-|t_m + t_n|^\alpha\right)$$

For any projection $J_k(x, \alpha)$ holds:

$$(J_k(x, \alpha) - J(x, \alpha)) \perp \exp(it_j x), \quad j = 1, 2, .., k$$

or

$$\langle (J_k(X, \alpha) - J(X, \alpha)), \exp(it_j X) \rangle = 0, \quad j = 1, 2, .., k$$

hence

$$\int_{-\infty}^{\infty} J_k(x, \alpha) p(x, \alpha) \exp(it_j x) dx = \int_{-\infty}^{\infty} J(x, \alpha) p(x, \alpha) \exp(it_j x) dx.$$

Hence, the calculation of both integrals yields:

$$\sum_{v=0}^{k} a_v \exp\left(-|t_v + t_j|^\alpha\right) = \exp\left(-|t_j|^\alpha\right) \cdot |t_j|^\alpha \cdot \ln |t_j|, \quad j = 0, 1, 2, .., k.$$

Notation

- $A(\alpha) = \{\mathrm{e}^{t_i + t_j}, i, j = 0, 1, 2, ..., k\}$,
- $b(\alpha) = (0, -|t_1|^\alpha \ln |t_1| e^{-|t_1|^\alpha}, .., -|t_k|^\alpha \ln |t_k| e^{-|t_k|^\alpha})^T$
- $\mathbf{t} = (0, t_1, t_2, ..., t_k)^T$
- $F(\mathbf{t}, x) = (1, 2\cos(t_1), 2\cos(t_2), ..., 2\cos(t_k))^T$

In this notation, $J_k(x, \alpha)$ can be represented as follows:

$$J_k(x, \alpha) = (A(\alpha))^{-1} b(\alpha) F(\mathbf{t}, x)$$

and if we have n observations $X_1, X_2, ..., X_n$ i.i.d. $\sim S_\alpha(1, 0, 0)$ then

$$\hat{\alpha} = \left\{ \alpha : (A(\alpha))^{-1} b(\alpha) \sum_{j=1}^{n} F(\mathbf{t}, X_j) = 0 \right\}.$$

The Fisher information will be calculated by means of the formula $I_k(\alpha) = E J_k^2(X, \alpha)$. According to [4] $I_k(\alpha) \to I(\alpha)$ for $\alpha \in (0, 2]$ as $k \to \infty$.

2.2 Estimation of the Q parameter.

$$J_k(\mathbf{x},\alpha) = \sum_{j=-k}^{k} a_j \exp\left(i \cdot \mathbf{t}_j^T \mathbf{x}\right)$$

where $a_j = a_{-j}$ and $\mathbf{t}_{-j} = -\mathbf{t}_j$, $\mathbf{t_j} \in R^d$ $j = 1, 2, ..., k$ and $\mathbf{t}_0 = \mathbf{0}$. Then

$$J_k(\mathbf{x},\alpha) = a_0 + 2\sum_{j=1}^{k} a_j \cos\left(\mathbf{t}_j^T \mathbf{x}\right)$$

$$\langle \exp(i \cdot \mathbf{t}_m^T \mathbf{X}), \exp(i \cdot \mathbf{t}_n^T \mathbf{X})\rangle = E\exp(i \cdot \mathbf{t}_m^T \mathbf{X}) \cdot \exp(i \cdot \mathbf{t}_n^T \mathbf{X}) = E\exp(i \cdot (\mathbf{t}_m + \mathbf{t}_n)^T \mathbf{X}) =$$
$$= \int_{\mathbf{R}^K} p(\mathbf{x},\alpha) \exp\left(i \cdot (\mathbf{t}_n + \mathbf{t}_m)^T \mathbf{x}\right) d\mathbf{x} = \exp\left(-\left|(\mathbf{t}_n + \mathbf{t}_m)^T \mathbf{Q} (\mathbf{t}_n + \mathbf{t}_m)\right|^{\frac{\alpha}{2}}\right)$$

Let us denote $J_k = J_k^{i*,j*}$ and $J = J^{i*,j*}$. For any projection $J_k(\mathbf{x},\mathbf{Q},\alpha)$ holds:

$$\left(J(\mathbf{x},\mathbf{Q},\alpha) - J_k(\mathbf{x},\mathbf{Q},\alpha)\right) \perp \exp\left(i \cdot \mathbf{t}_j^T \mathbf{x}\right)$$

Hence follows:

$$\int_{\mathbf{R}^K} \left(J(\mathbf{x},\mathbf{Q}\alpha) - J_k(\mathbf{x},\mathbf{Q},\alpha)\right) p(\mathbf{x},\alpha) \exp\left(i \cdot \mathbf{t}_j^T \mathbf{x}\right) d\mathbf{x} = 0, \quad j = 1, 2, ..., k$$

Hence

$$\int_{\mathbf{R}^K} \left(J(\mathbf{x},\mathbf{Q},\alpha)\right) p(\mathbf{x},\alpha) \exp\left(i \cdot \mathbf{t}_j^T \mathbf{x}\right) d\mathbf{x} = \int_{\mathbf{R}^K} \left(J_k(\mathbf{x},\mathbf{Q}\alpha)\right) p(\mathbf{x},\alpha) \exp\left(i \cdot \mathbf{t}_j^T \mathbf{x}\right) d\mathbf{x}$$

for j=1,2,...,k. Let us calculate both integrals separately:

$$\int_{\mathbf{R}^K} \left(J_k(\mathbf{x},\alpha)\right) p(\mathbf{x},\alpha) \exp\left(i \cdot \mathbf{t}_j^T \mathbf{x}\right) d\mathbf{x} = \int_{\mathbf{R}^K} \sum_{l=-k}^{k} a_l \exp\left(i \cdot \mathbf{t}_l^T \mathbf{x}\right) p(\mathbf{x},\alpha) \exp\left(i \cdot \mathbf{t}_j^T \mathbf{x}\right) d\mathbf{x} =$$

$$= \sum_{l=-k}^{k} \int_{\mathbf{R}^K} a_l \exp\left(i \cdot \mathbf{t}_l^T \mathbf{x}\right) p(\mathbf{x},\alpha) \exp\left(i \cdot \mathbf{t}_j^T \mathbf{x}\right) d\mathbf{x} = \sum_{l=-k}^{k} a_l \int_{\mathbf{R}^K} p(\mathbf{x},\alpha) \exp\left(i \cdot (\mathbf{t}_l^T + \mathbf{t}_j^T)\mathbf{x}\right) d\mathbf{x} =$$

$$= \sum_{l=-k}^{k} a_l \exp\left(-\left|(\mathbf{t}_l + \mathbf{t}_j)^T \mathbf{Q} (\mathbf{t}_l + \mathbf{t}_j)\right|^{\frac{\alpha}{2}}\right)$$

The second integral will be calculated as follows:

$$\int_{\mathbf{R}^K} \left(J(\mathbf{x},\mathbf{Q},\alpha)\right) p(\mathbf{x},\alpha) \exp\left(i \cdot \mathbf{t}_j^T \mathbf{x}\right) d\mathbf{x} = \int_{\mathbf{R}^K} \left(\frac{\left(\frac{\partial p(\mathbf{x},\alpha)}{\partial q_{i*,j*}}\right)}{p(\mathbf{x},\alpha)}\right) p(\mathbf{x},\alpha) \exp\left(i \cdot \mathbf{t}_j^T \mathbf{x}\right) d\mathbf{x}$$

$$= \int_{\mathbf{R}^K} \left(\frac{\partial p(\mathbf{x},\alpha)}{\partial q_{i*,j*}}\right) \exp\left(i \cdot \mathbf{t}_j^T \mathbf{x}\right) d\mathbf{x} = \frac{\partial}{\partial q_{i*,j*}} \int_{\mathbf{R}^K} p(\mathbf{x},\alpha) \exp\left(i \cdot \mathbf{t}_j^T \mathbf{x}\right) d\mathbf{x} =$$
$$= \frac{\partial}{\partial q_{i*,j*}} \exp\left(-\left|\mathbf{t}_j^T \mathbf{Q} \mathbf{t}_j\right|^{\frac{\alpha}{2}}\right) = -(t_j^T \mathbf{Q} t_j)^{\alpha/2-1} \exp\left(-\left|\mathbf{t}_j^T \mathbf{Q} \mathbf{t}_j\right|^{\frac{\alpha}{2}}\right) \frac{\alpha}{2} t_{i*} t_{j*}$$

Hence we will get the following sequence:

$$\sum_{l=-k}^{k} a_l \exp\left(-\left|(\mathbf{t}_l + \mathbf{t}_j)^T Q (\mathbf{t}_l + \mathbf{t}_j)\right|^{\frac{\alpha}{2}}\right) = -(\mathbf{t}_j^T \mathbf{Q} \mathbf{t}_j)^{\alpha/2} \exp\left(-\left|\mathbf{t}_j^T \mathbf{Q} \mathbf{t}_j\right|^{\frac{\alpha}{2}}\right) \frac{\alpha}{2} t_{i*} t_{j*}$$

In matrix form we get the following system of equations:

$$\begin{pmatrix} c_{0,0} & c_{0,1} & \dots & c_{0,k} \\ c_{1,0} & c_{1,1} & \dots & c_{1,k} \\ \dots & \dots & \dots & \dots \\ c_{k,0} & c_{k,1} & \dots & c_{k,k} \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ \dots \\ a_k \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ \dots \\ b_k \end{pmatrix}$$

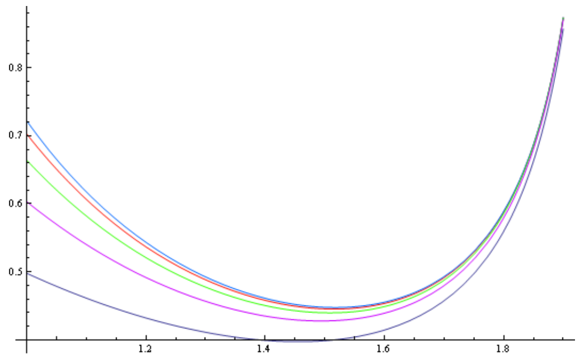

Figure 1: Graph of the Fisher information and the rate of convergence of $I_k(\alpha)$ to $I(\alpha)$ for $\alpha \in [1,2]$, for the blue line $k = 200$, for the rest k is lower.[6]

where $c_{m,l} = \exp\left(-\left|(\mathbf{t}_m + \mathbf{t}_l)^T \mathbf{Q}(\mathbf{t}_m + \mathbf{t}_l)\right|^{\frac{\alpha}{2}}\right)$ and $b_l = -(\mathbf{t}_l^T \mathbf{Q} \mathbf{t}_l)^{\alpha/2} \exp\left(-\left|\mathbf{t}_l^T \mathbf{Q} \mathbf{t}_l\right|^{\frac{\alpha}{2}}\right) \frac{\alpha}{2} t_{i*} t_{j*}$ where $m, l = 1, 2, ..., k$ Note that $c_{m,l} = c_{m,l}(\mathbf{Q})$ and $b_l = b_l(\mathbf{Q})$. α is supposed to be known, because we can determine it by analyzing univariate margins. Let us denote:

$$A(Q) = \{c_{i,j}, i, j = 0, 1, 2, .., k\}$$

So the coefficients will be determined as follows: $\mathbf{a}(\mathbf{Q}) = \mathbf{A}(\mathbf{Q})^{-1} \mathbf{b}(Q)$. Let us assume $\mathbf{t}_0 = \mathbf{0}$, i.e. $\mathbf{tt} = (\mathbf{0}, \mathbf{t}_1, \mathbf{t}_2, ..., \mathbf{t}_k)^T$ and denote

$$F(\mathbf{tt}, \mathbf{x}) = (2\cos(\mathbf{t}_0^T \mathbf{x}), 2\cos(\mathbf{t}_1^T \mathbf{x})..2\cos(\mathbf{t}_k^T \mathbf{x}))^T$$

$J_k(\mathbf{Q}) = \mathbf{a}(\mathbf{Q})^T F(\mathbf{tt}, \mathbf{x})$ and the estimate is: $\hat{\mathbf{Q}}_k = \{\mathbf{Q} : J_k(\mathbf{Q}) = 0\}$ This method gives us the estimate of the whole matrix $\mathbf{Q}$ although we we were chasing the estimate $q_{i*,j*}$ such that $\mathbf{Q} = \{q_{i,j} : i, j = 1, 2, .., d\}$. Therefore, we will take only the element $(\hat{q}_k)_{i*,j*}$ as the estimate of $q_{i*,j*}$ and having changed the indexes, continue the estimations of other elements of the matrix $\mathbf{Q}$.

3 Results

The **Table 1.** compares the quality of the estimation of the parameter α of the stable distribution $S_\alpha(1, 0, 0)$. There are compared two methods:

- CFB means the methodology based on characteristic function when we compare theoretical and empirical characteristic functions. This methodology is described in detail in [6] and the theory on which this methodology is based is described in [3] and [6].
- The methodology based on projections is symbolically denoted by MLP.

There were simulated the samples with 1000 elements from the stable distribution $S_\alpha(1, 0, 0)$. And then followed the estimation procedure by two just mentioned methodologies. From the table we can see that the quality of the MLP is much higher than the one of its counterpart. The value of k was chosen to be 150. The similar methodology can be applied to estimate two parameters (σ, α) of the stable distribution $S_\alpha(\sigma, 0, 0)$. Note that $\mathbf{Q}$ is a covariance matrix of some normal distribution. Therefore, finding $q_{i,i}$, $i = 1, 2, .., d$ enables us to deal with the correlation matrix. Moreover we will have that $q_{i,j} = q_{j,i}$, $i, j = 1, 2, .., d$. But simple algebraic operations lead to the conclusion that $q_{i,i} = \sigma_i$, i.e. $\sigma_{i,i}$ can be estimated in the first phase together with the α parameter. In other words it means that if we have to estimate the matrix $\mathbf{Q}_{d \times d}$ it does not mean that we have to estimate $d \times d$ parameters.

Conclusion

The aim of this paper is to present methods of operating with the multivariate elliptical stable distributions. By virtue of the Table 1. we showed that the methodology based on projections works even better then the methodology based on comparing empirical and theoretical characteristic functions and that such estimators converge to the MLP estimators. The calculations of the tail index was described

α	Estimator	Mean	Var	Mean$\pm 2\sigma$
1.1	MLP	1.11584	0.00056	[1.0684,1.1632]
	CFB	1.10255	0.00210	[1.0109,1.1942]
1.2	MLP	1.20275	0.00174	[1.1191,1.2863]
	CFB	1.20260	0.00220	[1.1087,1.2964]
1.3	MLP	1.29915	0.00213	[1.2067,1.3915]
	CFB	1.30260	0.00240	[1.2046,1.4005]
1.4	MLP	1.39966	0.00181	[1.3145,1.4847]
	CFB	1.40230	0.00260	[1.3003,1.5042]
1.5	MLP	1.49584	0.00206	[1.4049,1.5867]
	CFB	1.50310	0.00280	[1.3972,1.6089]
1.6	MLP	1.59134	0.00151	[1.5135,1.6691]
	CFB	1.60300	0.00290	[1.4953,1.7107]
1.7	MLP	1.70204	0.00187	[1.6154,1.7886]
	CFB	1.70120	0.00270	[1.5972,1.8051]
1.8	MLP	1.79173	0.00159	[1.7118,1.8717]
	CFB	1.80120	0.00240	[1.7032,1.8991]
1.9	MLP	1.89991	0.00113	[1.8326,1.9671]
	CFB	1.90220	0.00160	[1.8222,1.9822]

Table 1: Quality of the estimation. We compare mean, variance and $\mu \pm 2\sigma$ intervals to find out which estimator is more precise

in detail and the estimation of the dependence structure, i.e. $\mathbf{Q}$, can be conducted in a similar way. The ability to estimate the parameters of such distributions enables us to use them in multivariate models e.g. in some modifications of MGARCH [1] model and to extend the normal models with new properties making them closer to reality.

Acknowledgement.

This work was supported by the Grant Agency of the Czech Republic under grant GACR 402/09/H045

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